monadic λ-calculus

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概要

モナドは, 値を返す以外に特別な副作用をもつような関数を解釈するために導入された. 副作用としては状態を扱うもの, エラーを生成するもの, 非決定的なもの, などがある. ここではモナドを用いた semantics と, モナドを組み込んだ言語についてみていく.

目次

第 I 部 Monad

1　A Monad

定義 1 (monad)**.** $\mathbb{C}$ 上のモナドとは, $T : \mathbb{C} \to \mathbb{C}$ なる functor で, 次の構造をもつもののことである:

- 自然変換 $\eta : 1 \dot{\to} T$, $\mu : T^2 \dot{\to} T$ (それぞれ unit, multiplication などと呼ばれる)
- これらが図 1 の図式を満たす.

モナドとは, モノイダル圏 $([\mathbb{C}, \mathbb{C}], \circ, 1)$ のモノイド対象ともいえる. このとき η はモノイド構造の単位元を, μ はモノイド積を表している.

定義 2. T が $\mathbb{C}$ 上のモナド, $\mathbb{C}$ は product をもつとする. モナド T が **strong** とは, 自然変換 strength

$$
\begin{tikzcd}
T^3 \arrow[r, "\mu_T"] \arrow[d, "T_\mu"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta_T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T_\eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}
$$

図 1　monad laws

$\tau_{A,B} : A \times TB \to T(A \times B)$ (τ は A, B について natural) で図 2 の図式を満たすもののこと. ここで, associativity $(A \times B) \times C \xrightarrow{\cdot} A \times (B \times C)$ を γ とかいた.

$$
\begin{tikzcd}
1 \times TA \arrow[r, "\tau"] \arrow[dr, "\sim"] & T(1 \times A) \arrow[d, "\wr"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \times B) \times TC \arrow[r, "\tau"] \arrow[d, "\gamma"'] & T((A \times B) \times C) \arrow[dr, "T\gamma"] & \\
A \times (B \times TC) \arrow[r, "1\times\tau"] & A \times T(B \times C) \arrow[r, "\tau"] & T(A \times (B \times C))
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \times B \arrow[d, "1\times\eta"'] \arrow[dr, "\eta"] & & \\
A \times TB \arrow[r, "\tau"] & T(A \times B) & \\
A \times T^2B \arrow[u, "1\times\mu"] \arrow[r, "\tau"] & T(A \times TB) \arrow[r, "T\tau"] & T^2(A \times B) \arrow[ul, "\mu"']
\end{tikzcd}
$$

図 2　strength laws

モナドの例をいくつか挙げておこう. 以下の話で出てくるようなモナドは大体次の例に出てくるものと思ってもおそらく問題ない.

例 3. 代表的なモナドたち:

- partiality, option, maybe: $TA = 1 + A$; 失敗する可能性のある計算を表す
- exception: $T_{\mathrm{exn}}A = \mathrm{exn} + A$; 例外を発生させる (例外が起きた時専用の値を返す) 計算を表す
- list: $TA = \mu X.\, 1 + A \times X$; 非決定的な計算を表す
- state: $T_S A = S \to S \times A$; 型 S の状態をもつ計算を表す
- continuation: $T_R A = (A \to R) \to R$; 継続をもつ計算を表す

2　Monad in calculi

2.1　call-by-value

$\lambda^{\to}$ に product, 1, let-in を入れた call-by-value language λ を考える.

命題 4. $1 \to (-)$ は $\mathit{Syn}(\lambda)$ 上のモナドである.

Proof. functor $T : \mathrm{Syn} \to \mathrm{Syn}$ を, 次のように定義する:

- object: $TA = 1 \to A$
- arrow: $T(f : A \to B) = \lambda t^{1 \to A}.\, \lambda \star.\, f\,(t\,\star) : (1 \to A) \to (1 \to B)$

このとき, monad の operation は,

- unit: $\eta_A = \lambda a^A.\, \lambda\star.\, a : A \to (1 \to A)$
- join: $\mu_A = \lambda t^{1\to(1\to A)}.\, \lambda\star.\, t\star\star : (1 \to (1 \to A)) \to (1 \to A)$

によって定まる. さて, monad 則をみよう:

(assoc) $\mu \circ \mu_T = \mu \circ T\mu$ について,

$$
\begin{aligned}
(\mathrm{LHS}) &= \lambda M^{1\to1\to1\to A}.\, \mu(\mu\, M) \\
&= \lambda M.\, \mu\, (\texttt{let}\ m = M\ \texttt{in}\ \mu\, m) \\
&= \lambda M.\, \mu\, (\texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, m\star\star) \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \mu\, (\lambda\star.\, m\star\star) \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, (\lambda\star.\, m\star\star)\star\star \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, m\star\star\star \\
(\mathrm{RHS}) &= \lambda M^{1\to1\to1\to A}.\, \mu\, (T\mu\, M) \\
&= \lambda M.\, \mu\, (\texttt{let}\ m = M\ \texttt{in}\ T\mu\, m) \\
&= \lambda M.\, \mu\, (\texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, \mu\, (m\star)) \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \mu\, (\lambda\star.\, \mu\, (m\star)) \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, (\lambda\star.\, \mu\, (m\star))\star\star \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, \mu\, (m\star)\ \star \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, m\star\star\star
\end{aligned}
$$

となるので OK.

(left identity) $\mu \circ \eta_T = id$ について,

$$
\begin{aligned}
(\mathrm{LHS}) &= \lambda M^{1\to A}.\, \mu\, (\eta\, M) \\
&= \lambda M.\, \mu\, (\texttt{let}\ m = M\ \texttt{in}\ \eta\, m) \\
&= \lambda M.\, \mu\, (\texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, m) \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \mu\, (\lambda\star.\, m) \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, (\lambda\star.\, m)\star\star \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ \lambda\star.\, m\star \\
&= \lambda M.\, \texttt{let}\ m = M\ \texttt{in}\ m \\
&= \lambda M.\, M \\
&= (\mathrm{RHS})
\end{aligned}
$$

よって OK. (right identity) も同様. □

call-by-value なので, MN は M を評価, N を評価, 最後に代入の順番で行うことに注意.

第 II 部

Monadic Metalanguage

3 Metalanguage

3.1 Syntax

定義 5. $\lambda^{\rightarrow}$ を, 次のように拡張したものを, ここでは λ^T とよぶことにする.

$$\begin{aligned} \alpha &::= \ldots \mid T\alpha \\ M &::= \ldots \mid [M] \mid \mathtt{let}\ x \leftarrow M_1\ \mathtt{be}\ M_2 \end{aligned}$$

また, typing rule は図 3 の通り.

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash [M] : TA}\ (\textsc{unit}) \qquad\qquad \frac{\Gamma \vdash M_1 : TA \qquad \Gamma, x : A \vdash M_2 : TB}{\Gamma \vdash \mathtt{let}\ x \leftarrow M_1\ \mathtt{be}\ M_2 : TB}\ (\textsc{let})$$

図 3 Typing rules of λ^T

λ^T での T は, computational effect を表す. つまり, 型 TA をもつプログラムは, A の computational effect を表している. $[-]$ は通常の意味での値を computational effect に送り, $\mathtt{let}\ldots\mathtt{be}\ldots$ は effect つきの関数を effect つきの値に適用する operation である ($TA \to (A \to TB) \to TB$). 実際に, T を具体的に与えた言語を考える際は T によって特別な operator がさらに追加されることがある.

3.2 Equational Theory

λ^T の equational theory は, $\lambda^{\rightarrow}$ に次を加えたものとする.

$$\mathtt{let}\ y \leftarrow (\mathtt{let}\ x \leftarrow M_1\ \mathtt{be}\ M_2)\ \mathtt{be}\ M_3 = \mathtt{let}\ x \leftarrow M_1\ \mathtt{be}\ \mathtt{let}\ y \leftarrow M_2\ \mathtt{be}\ M_3 \tag{assoc}$$

$$\mathtt{let}\ x \leftarrow [M]\ \mathtt{be}\ N = N[M/x] \tag{$T.\beta$}$$

$$\mathtt{let}\ x \leftarrow M\ \mathtt{be}\ [x] = M \tag{$T.\eta$}$$

4 Monadic semantics

λ^T の categorical semantics を考える. λ^T の T の semantics はモナドで与えられる.

4.1 Syntactic category

命題 6. λ^T の *syntactic category* はモナド T をもつ.

Proof. ただし, モナド T の構造は次のようにして与える.

$$\begin{aligned} T(A) &= TA \\ \eta_A &= [x : A \vdash [x] : TA] \\ \mu_A &= [x : TTA \vdash \mathtt{let}\ y \leftarrow x\ \mathtt{be}\ y : TA] \end{aligned}$$

これがモナドを与えることを示そう.

(T の mapping)　T が射をどう map するかも書いておこう．$T(f : A \to B) = [x : TA \vdash \mathtt{let}\ y \leftarrow x\ \mathtt{be}\ [f(y)] : TB]$ で与える．これが functor になることは明らかであろう．

(η の naturality)　$[y : A \vdash t : B]$ に対し，$[x : A \vdash \mathtt{let}\ y \leftarrow [x]\ \mathtt{be}\ [t] : TB] = [x : A \vdash [z]\,[(t[x/y])/z] : TB]$ を示せばよい．

$$\begin{aligned}
(\mathrm{LHS}) &= [x : A \vdash \mathtt{let}\ y \leftarrow [x]\ \mathtt{be}\ [t] : TB] \\
&= [x : A \vdash [t]\,[x/y] : TB] && (T.\beta) \\
&= [x : A \vdash [t[x/y]] : TB] \\
&= (\mathrm{RHS})
\end{aligned}$$

によって OK.

(μ の naturality)　$[z : A \vdash t : B]$ に対し，$[x : TTA \vdash \mathtt{let}\ z \leftarrow (\mathtt{let}\ y \leftarrow x\ \mathtt{be}\ y)\ \mathtt{be}\ [t] : TA] = [x : TTA \vdash \mathtt{let}\ y \leftarrow (\mathtt{let}\ z_1 \leftarrow x\ \mathtt{be}\ [\mathtt{let}\ z \leftarrow z_1\ \mathtt{be}\ [t]])\ \mathtt{be}\ y : TA]$ を示せばよい．

$$\begin{aligned}
(\mathrm{LHS}) &= [x : TTA \vdash \mathtt{let}\ y \leftarrow x\ \mathtt{be}\ \mathtt{let}\ z \leftarrow y\ \mathtt{be}\ [t] : TA] && (\mathrm{assoc}) \\
(\mathrm{RHS}) &= [x : TTA \vdash \mathtt{let}\ z_1 \leftarrow x\ \mathtt{be}\ \mathtt{let}\ y \leftarrow [\mathtt{let}\ z \leftarrow z_1\ \mathtt{be}\ [t]]\ \mathtt{be}\ y : TA] && (\mathrm{assoc}) \\
&= [x : TTA \vdash \mathtt{let}\ z_1 \leftarrow x\ \mathtt{be}\ \mathtt{let}\ z \leftarrow z_1\ \mathtt{be}\ [t] : TA] && (T.\beta)
\end{aligned}$$

により OK.

(μ の associativity)　$\mu \circ \mu_T = \mu \circ T\mu$ を示せばよい．

$$\begin{aligned}
(\mathrm{LHS}) &= [x : TTTA \vdash \mathtt{let}\ y \leftarrow x\ \mathtt{be}\ \mathtt{let}\ z \leftarrow y\ \mathtt{be}\ z : TA] \\
(\mathrm{RHS}) &= [x : TTTA \vdash \mathtt{let}\ z \leftarrow (\mathtt{let}\ x' \leftarrow x\ \mathtt{be}\ [\mathtt{let}\ y \leftarrow x'\ \mathtt{be}\ y])\ \mathtt{be}\ z : TA] \\
&= [x : TTTA \vdash \mathtt{let}\ x' \leftarrow x\ \mathtt{be}\ \mathtt{let}\ z \leftarrow [\mathtt{let}\ y \leftarrow x'\ \mathtt{be}\ y]\ \mathtt{be}\ z : TA] && (\mathrm{assoc}) \\
&= [x : TTTA \vdash \mathtt{let}\ x' \leftarrow x\ \mathtt{be}\ \mathtt{let}\ y \leftarrow x'\ \mathtt{be}\ y : TA] && (T.\beta)
\end{aligned}$$

により OK.

($\mu - \eta$ の left identity)　$\mu \circ \eta T = id$ を示せばよい．

$$\begin{aligned}
(\mathrm{LHS}) &= [x : TA \vdash \mathtt{let}\ y \leftarrow [x]\ \mathtt{be}\ y : TA] \\
&= [x : TA \vdash x : TA] && (T.\beta) \\
&= (\mathrm{RHS})
\end{aligned}$$

($\mu - \eta$ の right identity)　$\mu \circ T\eta = id$ を示せばよい．

$$\begin{aligned}
(\mathrm{LHS}) &= [x : TA \vdash \mathtt{let}\ y \leftarrow (\mathtt{let}\ x' \leftarrow x\ \mathtt{be}\ [[x']])\ \mathtt{be}\ y : TA] \\
&= [x : TA \vdash \mathtt{let}\ y \leftarrow [x]\ \mathtt{be}\ y : TA] && (T.\eta) \\
&= [x : TA \vdash x : TA] && (T.\beta) \\
&= (\mathrm{RHS})
\end{aligned}$$

以上により，T はモナドを与える．　□

命題 7. λ^T のモナド T は *strong* である．

Proof. strength $\tau : A \times TB \to T(A \times B)$ は，$[x : A, y : TB \vdash \mathtt{let}\ y' \leftarrow y\ \mathtt{be}\ [\langle x, y' \rangle] : T(A \times B)]$ によって与える．

(τ の naturality)　B について．$[z : B \vdash t : B']$ に対し，

$$T(1 \times t) \circ \tau_{A,B} = \tau_{A,B'} \circ (1 \times Tt)$$

を示す．これは,

$$
\begin{aligned}
\text{(LHS)} &= [x:A, y:TB \vdash \texttt{let}\ \langle w,z\rangle \leftarrow (\texttt{let}\ y' \leftarrow y\ \texttt{be}\ [\langle x,y'\rangle])\ \texttt{be}\ [\langle w,t\rangle] : T(A\times B')] \\
&= [x:A, y:TB \vdash \texttt{let}\ y' \leftarrow y\ \texttt{be}\ \texttt{let}\ \langle w,z\rangle \leftarrow [\langle x,y'\rangle]\ \texttt{be}\ [\langle w,t\rangle] : T(A\times B')] && \text{(assoc)} \\
&= [x:A, y:TB \vdash \texttt{let}\ y' \leftarrow y\ \texttt{be}\ [\langle x, t[y'/z]\rangle] : T(A\times B')] && (T.\beta) \\
&= [x:A, y:TB \vdash \texttt{let}\ z \leftarrow y\ \texttt{be}\ [\langle x,t\rangle] : T(A\times B')] \\
\text{(RHS)} &= [x:A, y:B \vdash \texttt{let}\ y' \leftarrow (\texttt{let}\ z \leftarrow y\ \texttt{be}\ [t])\ \texttt{be}\ [\langle x,y'\rangle] : T(A\times B')] \\
&= [x:A, y:B \vdash \texttt{let}\ z \leftarrow y\ \texttt{be}\ \texttt{let}\ y' \leftarrow [t]\ \texttt{be}\ [\langle x,y'\rangle] : T(A\times B')] && \text{(assoc)} \\
&= [x:A, y:B \vdash \texttt{let}\ z \leftarrow y\ \texttt{be}\ [\langle x,t\rangle] : T(A\times B')] && (T.\beta)
\end{aligned}
$$

により OK. A については明らか.

(τ と left identity の compatibility)

$$
\begin{aligned}
&[x:TA \vdash \texttt{let}\ \langle\star,x'\rangle \leftarrow \tau\langle\star,x\rangle\ \texttt{be}\ [x'] : TA] \\
&= [x:TA \vdash \texttt{let}\ \langle\star,x'\rangle \leftarrow (\texttt{let}\ y' \leftarrow x\ \texttt{be}\ [\langle\star,y'\rangle])\ \texttt{be}\ [x'] : TA] \\
&= [x:TA \vdash \texttt{let}\ y' \leftarrow x\ \texttt{be}\ \texttt{let}\ \langle\star,x'\rangle \leftarrow [\langle\star,y'\rangle]\ \texttt{be}\ [x'] : TA] \\
&= [x:TA \vdash \texttt{let}\ y' \leftarrow x\ \texttt{be}\ [y'] : TA] \\
&= [x:TA \vdash x : TA]
\end{aligned}
$$

により, OK.

(τ と associativity)

$$
\begin{aligned}
(\text{上}) &= [x:A, y:B, t_c:TC \vdash T\gamma(\tau\langle\langle x,y\rangle, t_c\rangle) : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash T\gamma(\texttt{let}\ z \leftarrow t_c\ \texttt{be}\ [\langle\langle x,y\rangle, z\rangle]) : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash \texttt{let}\ \langle\langle x',y'\rangle, z'\rangle \leftarrow (\texttt{let}\ z \leftarrow t_c\ \texttt{be}\ [[\langle\langle x,y\rangle, z\rangle]])\ \texttt{be}\ [\langle x', \langle y',z'\rangle\rangle] : \ldots] \\
&= [x:A, y:B, t_c:TC \vdash \texttt{let}\ z \leftarrow t_c\ \texttt{be}\ \texttt{let}\ \langle\langle x',y'\rangle, z'\rangle \leftarrow [[\langle\langle x,y\rangle, z\rangle]]\ \texttt{be}\ [\langle x', \langle y',z'\rangle\rangle] : \ldots] \\
&= [x:A, y:B, t_c:TC \vdash \texttt{let}\ z \leftarrow t_c\ \texttt{be}\ [\langle x, \langle y,z\rangle\rangle] : T(A\times(B\times C))] \\
(\text{下}) &= [x:A, y:B, t_c:TC \vdash \tau((1\times\tau)(\gamma\langle\langle x,y\rangle, t_c\rangle)) : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash \tau((1\times\tau)\langle x, \langle y, t_c\rangle\rangle : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash \tau\langle x, \texttt{let}\ z' \leftarrow t_c\ \texttt{be}\ [\langle y,z'\rangle]\rangle : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash \texttt{let}\ p \leftarrow (\texttt{let}\ z' \leftarrow t_c\ \texttt{be}\ [\langle y,z'\rangle])\ \texttt{be}\ [\langle x,p\rangle] : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash \texttt{let}\ z' \leftarrow t_c\ \texttt{be}\ \texttt{let}\ p \leftarrow [\langle y,z'\rangle]\ \texttt{be}\ [\langle x,p\rangle] : T(A\times(B\times C))] \\
&= [x:A, y:B, t_c:TC \vdash \texttt{let}\ z' \leftarrow t_c\ \texttt{be}\ [\langle x, \langle y,z'\rangle\rangle] : T(A\times(B\times C))]
\end{aligned}
$$

より OK.

(τ と η)

$$
\begin{aligned}
[x:A, y:B \vdash \tau\langle x, [y]\rangle : T(A\times B)] &= [x:A, y:B \vdash \texttt{let}\ y' \leftarrow [y]\ \texttt{be}\ [\langle x,y'\rangle] : T(A\times B)] \\
&= [x:A, y:B \vdash [\langle x,y\rangle] : T(A\times B)] \\
&= [x:A, y:B \vdash \eta\langle x,y\rangle : T(A\times B)]
\end{aligned}
$$

より OK.

(τ と μ)

$$
\begin{aligned}
(\text{上}) &= [a : A, b'' : T^2B \vdash \tau\,\langle a, \mu b''\rangle : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \texttt{let}\ b \leftarrow \mu b''\ \texttt{be}\ [\langle a, b\rangle] : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \texttt{let}\ b \leftarrow \texttt{let}\ y \leftarrow b''\ \texttt{be}\ y\ \texttt{be}\ [\langle a, b\rangle] : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \texttt{let}\ b' \leftarrow b''\ \texttt{be}\ \texttt{let}\ b \leftarrow b'\ \texttt{be}\ [\langle a, b\rangle] : T(A \times B)] \\
(\text{下}) &= [a : A, b'' : T^2B \vdash \mu(T\tau(\tau\,\langle a, b''\rangle)) : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \mu(T\tau(\texttt{let}\ b' \leftarrow b''\ \texttt{be}\ [\langle a, b'\rangle])) : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \mu(\texttt{let}\ b' \leftarrow b''\ \texttt{be}\ \texttt{let}\ b \leftarrow b'\ \texttt{be}\ [[\langle a, b\rangle]]) : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \texttt{let}\ b' \leftarrow b''\ \texttt{be}\ \texttt{let}\ b \leftarrow b'\ \texttt{be}\ \texttt{let}\ y \leftarrow [[\langle a, b\rangle]]\ \texttt{be}\ y : T(A \times B)] \\
&= [a : A, b'' : T^2B \vdash \texttt{let}\ b' \leftarrow b''\ \texttt{be}\ \texttt{let}\ b \leftarrow b'\ \texttt{be}\ [\langle a, b\rangle] : T(A \times B)]
\end{aligned}
$$

により OK.

以上より, τ は strength である. □

参考文献

[1] E. Moggi. 1991. “Notions of computation and monads". Inf. Comput. 93, 1 (July 1991), 55-92.

[2] Carsten Führmann. 2000. “The structure of call-by-value". Doctor of Philosophy, University of Edinburgh.